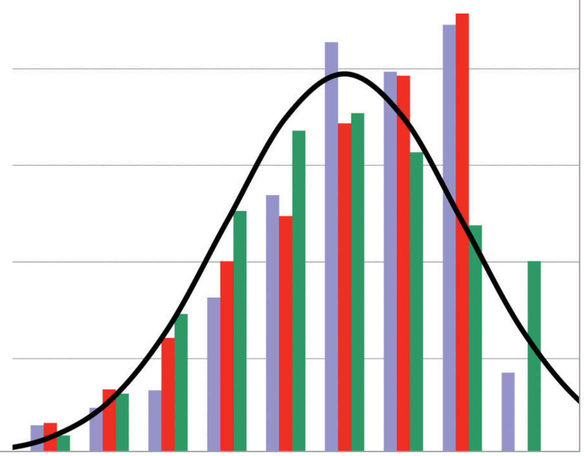
October 2013

ACT Research Report Series

2013 (7)



A Comparison of Three Methods for Computing Scale Score Conditional Standard Errors of Measurement

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A Comparison of Three Methods for Computing Scale Score Conditional Standard Errors of Measurement

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Abstract

Professional standards for educational testing recommend that both the overall standard error of measurement and the conditional standard error of measurement (CSEM) be computed on the score scale used to report scores to examinees. Several methods have been developed to compute scale score CSEMs. This paper compares three methods, based on classical test theory, item response theory, and the four-parameter beta compound binomial model. The three methods are compared using data from a single form of the ACT® College Readiness Assessment. The results indicate that all three methods produce comparable results.

A Comparison of Three Methods for Computing Scale Score Conditional Standard Errors of Measurement

Introduction

Interpreting scores from educational tests requires considering the scores' precision (the extent to which scores would be replicable on repeated testing of the same examinees with a parallel instrument on the same measurement occasion). Professional testing standards have long directed that reported scores (henceforth, "scale scores") should be accompanied by estimates of their precision, such as the standard error of measurement, at various points along the score scale (American Educational Research Association, American Psychological Association, & National Council on Measurement in Education, 1999). The conditional standard error of measurement (CSEM) is the standard deviation of an examinee's observed score that would be expected over repeated parallel measurements of an examinee with a fixed, unchanging true score. Although procedures for computing the CSEM on the raw number-correct score scale have existed since at least the 1940s, the development of appropriate estimators for the CSEM on reported score scales obtained by nonlinear transformations of raw scores is more recent, and few studies have compared the performance of these estimators. The purpose of this paper is to illustrate two procedures for computing scale score CSEM using the delta method (Feldt & Qualls, 1998; Lord, 1980), and to compare the CSEMs computed by these procedures with CSEM estimates from a strong true score theory-based method (Kolen, Hanson, & Brennan 1992; Lord, 1965) that is used by the ACT® College Readiness Assessment (ACT, 2007). Real data from a single form of the ACT test, comprised of multiple-choice tests in English, Mathematics, Reading, and Science, are used for the illustration and comparison.

Several methods for estimating the exact or approximate number-correct true score CSEM, based on different models for observed test score data, have been proposed: analysis of variance (Brennan, 1998; Jarjoura, 1986), four-parameter beta compound binomial (Lord, 1965), item response theory (IRT) (Lord, 1980), binomial error (Lord, 1957), compound binomial error (Feldt, 1984), and classically-parallel tests (Thorndike, 1951; Woodruff, 1991). All of these methods can be directly extended to computation of CSEMs for linear transformations of number-correct raw scores. In practice though, scale scores are often nonlinear transformations of raw scores. In addition, particularly when scale scores on several forms must be equated, the final raw-to-scale score conversion for a certain form is often expressed as a table of corresponding values instead of as a function.

To approximate the squared scale score CSEM when the raw-to-scale score transformation is a known, possibly nonlinear, continuous function, application of the delta method for computing the sampling variance of a function of a statistic can be used (Lord, 1980). The delta method (Kendall & Stuart, 1977) can be simply stated. Let *X* denote a random variable with population mean μ_X and population variance σ_X^2, and let $Y = g(X)$ be a function of *X*. Then an approximate value for the variance of $g(X)$ may be computed as follows:

$$\sigma_Y{}^2 = \left(\frac{dg(X)}{dX} \right)^2_{X=\mu_X} \sigma_X{}^2 .$$

The subscript $X = \mu_X$ on the derivative of $g(X)$ denotes that the derivative should be evaluated at the value μ_X. For illustration, consider a sample of *N* observations on the random variable *X* and let m_X and s_X^2 denote the sample mean and variance of *X*. Note that m_X is a random variable with mean $\mu_{m_x} = \mu_x$ and variance $\sigma_{m_x}^2 = \sigma_x^2 / N$. Consider the function of the sample mean, $Y = 3m_X^4$, the derivative of which is $12m_X^3$. An approximate formula for the variance of this function is

$$\sigma_Y^2 = (12\mu_{m_X}^3)^2 \sigma_{m_X}^2 = 144\mu_x^6 \sigma_x^2 / N .$$

Because m_X is an estimate of its own mean, an estimate for the approximate variance of Y can be computed by replacing the parameters in the preceding equation by their sample estimates which yields

$$s_Y^2 = 144 m_x^6 s_x^2 / N .$$

In practical measurement situations, especially if integer-valued scale scores are required for reporting, the transformation of raw to scale scores is seldom a continuous function, but instead, a score conversion table specifying a many-to-one conversion. Using the delta method in this situation requires approximating the tabled values by a known function, then differentiating that function to obtain its slope at particular raw score values. A rough approximation for the slope at a certain raw score value could be found by substitution of distinct, adjacent integer scale score values into the definitional equation for finding a slope between two points (e.g., Kolen & Brennan, 1995). Seeking a more precise and stable solution, Feldt and Qualls (1998) suggested fitting a high-degree polynomial function to the conversion table points. Cubic spline functions, also relatively easy to differentiate, have been proposed as an alternative to fit the conversion table values (Feldt & Qualls, 1998). In this study, polynomial functions will be used to approximate the raw-to-scale score conversion because they use data from the entire scale score distribution and are readily computed by many software packages. They also produce smoother estimated measurement error variance distributions than cubic spline approximation, desirable because population error variance functions are presumed to be smooth (Lee, Brennan, & Kolen, 1998).

In principle, any method for estimating the raw score CSEM could be combined with the delta method to estimate the scale score CSEM. Lord (1980) described how to use the delta method in conjunction with an IRT model to calculate the scale score CSEM when scale scores are a known continuous function of an observed score. Feldt and Qualls (1998) suggested application of the delta method employing Feldt's (1984) raw score CSEM estimate for tests consisting of distinct content strata. This paper will focus on deriving CSEMs appropriate for scale scores that are a transformation of number-correct raw scores, as did Feldt and Qualls.

Estimating Scale Score CSEM under a Classical Test Theory Model

Under classical test theory (CTT), the total raw score for a test composed of k items can be represented by the decomposition $X = T + E$, where X denotes an examinee's observed number-correct raw score, T an examinee's true score, and E denotes the error score for the examinee. No specific distributional assumptions about the measurement errors or true scores are implied by the CTT decomposition. The true and error score are assumed to be uncorrelated but dependent because the variance of E is allowed to vary for different values of T. The squared conditional standard error of measurement on the raw score scale is defined as the conditional variance of X given T and denoted $\sigma^2(X \mid T)$. If $Y = g(X)$ denotes the raw-to-scale-score conversion, then the scale score squared CSEM is similarly defined as the conditional variance of the observed scale score Y given T, denoted $\sigma^2(g(X) \mid T)$. Let t_o denote a fixed specific value of the true score T. Then, since the expectation of X given t_o is t_o and t_o is a constant, the delta method approximation of $\sigma^2(g(X) \mid T)$ is given by

$$\begin{aligned}
\sigma^2(g(X) \mid T = t_o) &= \left(\frac{dg(X)}{dX}\right)^2_{X=t_o} \sigma^2(X \mid T = t_o) \\
&= \left(\frac{dg(X)}{dX}\right)^2_{X=t_o} \sigma^2(t_o + E \mid T = t_o) \qquad (1) \\
&= \left(\frac{dg(X)}{dX}\right)^2_{X=t_o} \sigma^2(E \mid T = t_o).
\end{aligned}$$

Practical application of the method requires a sample estimate for $\sigma^2(X \mid T = t_o)$. Feldt, Steffen, and Gupta (1985) discuss several methods for estimating $\sigma^2(X \mid T = t_o)$. The simplest way, based on a CTT model, is Thorndike's (1951) difference method, which depends on splitting the test into two classically-parallel half-tests, X_1 and X_2, with identical score distributions. Then, using the difference method, $\sigma^2(X \mid T)$ is estimated by

$$s^2(X \mid T = t_0) \approx s^2(X_1 - X_2 \mid X = t_0). \tag{2}$$

Note that this equation is an approximation because the left side conditions on *T* whereas the right side conditions on *X*. (See Woodruff (1990) for a discussion of this issue.) Combining Equations (1) and (2) yields

$$s^2(g(X) \mid T = t_o) \approx \left(\frac{dg(X)}{dX}\right)^2_{X=t_o} s^2(X_1 - X_2 \mid X = t_o). \tag{3}$$

Equation (3) defines a CTT estimate for the scale score measurement error variance, the squared CSEM, by letting t_o take the value of each of possible number correct raw score 0, 1, 2, … , *k* for a test with *k* items.

Use of the Thorndike difference method to estimate conditional raw score error variance requires the assumption that the parallel tests, used in the theoretical repeated measurements of an examinee that defines the CSEM, be divisible into parallel half-tests. No assumptions are made about the distribution of measurement errors in the examinee population. The delta method requires the assumptions that the polynomial selected is a suitable approximation to the raw-to-scale score transformation function and a first-order Taylor series expansion accurately approximates the polynomial function at all raw score values.

Estimating Scale Score CSEM under an Item Response Theory Model

Alternatively, an IRT model can be fit to the item response data and the error variance conditional on the IRT θ scale score can be estimated from the IRT item parameter estimates. IRT models assume that conditional on θ, measurement error follows a compound or generalized binomial distribution (Lord, 1980, p 85). Because it is often the case in practice that the scale scores reported to examinees are a function of number-correct raw scores even when IRT is used for some aspects of a testing program such as scaling, linking, or equating, this study considers the situation in which the reporting scale score is a function of the number-correct raw score rather than the estimated IRT theta scale score. Following Lord (1980), let $\xi = \xi(\theta) = \sum_{i=1}^{k} P_i(\theta_0)$ denote number-correct true score as a function of θ and let $P(\theta)$ denote the item characteristic curve for a particular IRT model applied to a test with k items. Then the conditional expected value of an observed raw score given a true raw score is an increasing function of θ given by

$$E(X \mid \xi = \xi_0) = E(X \mid \xi = \xi(\theta_0)) = E(X \mid \theta = \theta_0) = \sum_{i=1}^{k} P_i(\theta_0), \tag{4}$$

where ξ_0 and θ_0 denote specified corresponding values. It also follows from the assumption of local independence that the conditional variance of observed raw score given true raw score is

$$\sigma^2(X \mid \xi = \xi_0) = \sigma^2(X \mid \xi = \xi(\theta_0)) = \sigma^2(X \mid \theta = \theta_0) = \sum_{i=1}^{k} P_i(\theta_0)(1 - P_i(\theta_0)). \tag{5}$$

Equations (4) and (5) are taken from Lord (1980, p 85). They assert that the values of the total raw score conditional mean and variance do not change when you condition them on a different true score metric. This is in contrast to the information function which does change. Of course, if you transform the total raw score then their values do change. For raw score transformation $g(X)$, inserting equations (4) and (5) into the general equation for the delta method approximation and employing estimated item parameter values yields the following estimate for the scale score squared CSEM:

$$s^2(g(X) \mid \xi_0 = \xi(\theta_0)) = \left(\frac{dg(X)}{dX}\right)^2_{X=\xi_0} \sum_{i=1}^{k} \hat{P}_i(\theta_0)(1-\hat{P}_i(\theta_0)), \tag{6}$$

where the derivative is evaluated at the true raw score value $\xi_0 = \xi(\theta_0)$. Equations (6) is essentially an application of Lord's (1980, p 79) equation (5-22). One way to apply the method in practice is to solve numerically the equation $\xi = \xi(\theta)$ for all (ξ_0, θ_0) pairs where ξ_0 takes on the integer values 0, 1, 2, …, *k*. A Newton Raphson method was used to accomplish this in the IRT examples given later in this paper. Alternatively, one could just compute $\xi(\theta)$ for a grid of closely spaced θ points and then use Equation 6 to compute the scale score CSEM at all of the θ points. An appropriately modified version of this alternative procedure is used in the third method presented in this paper.

Between the two scale score CSEM estimation methods discussed so far, the IRT variant of the delta method requires the strongest assumptions. In addition to requiring the assumptions for a delta method approximation, use of the estimator in Equation 6 requires the common IRT assumptions of unidimensionality of item response data and conditional independence of item error scores. In addition, if a three parameter IRT model is used with a guessing parameter, denoted by *C*, then

$$\xi(\theta_0) = \sum_{i=1}^{k} P_i(\theta_0) \geq \sum_{i=1}^{k} C_i \tag{7}$$

for all values of θ. Consequently, this method cannot be used to compute the CSEM for true scores values less than the sum of the *C*s, or what could be called the estimated true guessing score on the test.

Estimating Scale Score CSEM under a Beta Compound Binomial Model

The third method considered for estimating scale score CSEM is one presented by Lord (1965) and further developed by Kolen et al. (1992). It is based on a strong true score theory model, the four-parameter beta compound binomial model (BCBM). Strong true score theory models, including both the beta compound binomial model and IRT models, adopt assumptions regarding the form of the conditional distributional of error and, in some instances, the marginal distribution of true scores, rendering the CTT decomposition of observed scores empirically testable. Let τ denote an examinee's true proportion correct score. A general expression for a strong true score theory model for some infinite population of examinees can be given as

$$\Pr(X = i) = \int_0^1 \Pr(X = i | \tau) h(\tau) d\tau , \qquad (8)$$

where $\Pr(X = i)$ is the marginal probability of a raw score i, $\Pr(X = i \mid \tau)$ is the conditional probability of raw score i given true score τ, and $h(\tau)$ is the marginal density of true scores. Equation 8 describes a general class of strong true score theory models that posit particular distributions for both true and error scores including, but not limited to, the four-parameter BCBM.

In applying a BCBM to equation (8), Kolen et al. (1992) assume that a total test is divided into several subtests they call strata. Within each stratum the conditional probability for that stratum's raw score is assumed to be binomial given a within stratum proportion correct true score. Hence, the integral in equation (8) becomes a multiple integral over a product of binomial distributions groups of which depend on different proportion correct true scores, $\tau_1, \tau_2, ..., \tau_k$. However, Kolen et al. then fit Lord's (1965) two-term approximation of the compound binomial distribution to the conditional probability distribution of the total test raw score so that this conditional probability distribution depends only on a single proportion correct true score rather than on all the proportion correct true scores for all the strata. They thereby reduce their model to equation (8) with just a single integral. This single proportion correct true score is assumed to follow a four-parameter beta distribution, with two parameters specifying the shape of the score distribution, which is quite flexible, and two limiting parameters between 0 and 1 bounding the distribution. Because Lord's approximation to the compound binomial distribution requires an estimate of the average error variance for the total test score, consistent with the assumptions of a binomial distribution within each stratum, Feldt's (1984) stratified binomial error variance coefficient is used to compute the average total test score error variance. Estimation of the four-parameter BCBM by the method of moments is discussed in Lord (1965) and Hanson (1991).

Under the Kolen et al. (1992) stratified compound binomial model, measurement error is held to occur due to sampling of items for a given test form from a stratified domain of items, a process characterizing test development from a table of specifications, where stratification is based on item content. Different forms of the test are taken to be randomly parallel across corresponding strata in repeated sampling; that is, items assembled to create a particular form's strata are held to represent a random sample from the domain for each stratum, although no specific criterion for form equity is invoked. Measurement errors for examinees with a particular set of stratum true scores are assumed to be independent and follow a binomial distribution within each stratum, and to be independent across strata. Within each stratum, the assumption of a binomial distribution given a particular true score value usually requires that all items be equally difficult for all examinees with the same true score value. However, various assumptions about the administration of randomly sampled items to randomly sampled examinees are sometimes used to justify a binomial model even when items have unequal difficulties, but such assumptions tend to complicate the definition of error variance (Lord & Novick, 1968). This paper is only interested in comparing the CSEM results from the three models and not justifying any one model.

Given a particular strong true score theory model for observed number-correct scores conforming to the general expression in Equation 8, and a certain transformation of number-correct raw scores, $g(i)$, permitted to be nonlinear, Kolen et. al. (1992) define the conditional measurement error variance for scale scores as

$$\begin{aligned}\sigma^2(g(X)\,|\,\tau) &= E\{[g(X)-E(g(X\,|\,\tau))]^2\,|\,\tau\} \\ &= \sum_{i=0}^{k}\left\{\left(g(i)-\left[\sum_{i=0}^{k} g(i)\Pr(X=i\,|\,\tau)\right]\right)^2 \middle|\, \tau\right\}\Pr(X=i\,|\,\tau)\end{aligned} \tag{9}$$

where, as before, E denotes the expectation operator and all other notation is as given previously. For different values of true scores, estimates of conditional probabilities for the four-parameter BCBM are substituted into Equation 8 to yield the estimated squared scale score CSEM, $s^2(g(X)\,|\,\tau)$. Unlike the CTT and IRT variants of the delta method outlined previously, which approximate the raw score transformation function by a polynomial, this BCBM method for CSEM estimation uses the exact transformation to compute the scale score CSEM.

Synopsis of Results from Previous Studies

In addition to the scale score CSEM estimation methods outlined above: the two delta methods (CTT and IRT) with polynomial approximation to the raw-to-scale score conversion table and the four-parameter beta compound binomial method, at least four other distinct methods have been proposed. Because calculation of a polynomial regression was believed to be beyond the capabilities of some test developers, Feldt and Qualls (1998) suggested a method, typically referenced as the "Feldt-Qualls procedure," that uses a simpler approximation to the derivative of the polynomial conversion in the delta method scale score CSEM estimation equation. Details are provided in Feldt and Qualls (1998). Although the Feldt-Qualls procedure has been rendered to some degree obsolete by the relative ease of computing polynomial functions in many statistical software packages, it has been tested repeatedly in comparison studies. Kolen, Zeng, and Hanson (1996) demonstrated that the general formula for the scale score CSEM under a strong true score theory model (Kolen et al., 1992) is also applicable when an IRT model, rather than a beta compound binomial model, is assumed for conditional number-correct score frequencies, provided a particular form for the population theta score distribution is given. Additionally, methods for raw score CSEM estimation assuming a binomial or compound binomial model for measurement error have been directly extended to treat scale scores (Brennan & Lee, 1999).

Demonstrations of the proposed CSEM estimation techniques have warranted several general conclusions regarding the pattern and magnitude of scale score CSEMs. When scale score type is varied, e.g., grade-equivalent scores versus percentile ranks, etc., the shape of the estimated CSEM function also varies with each different transformation of raw scores from a given test potentially yielding a unique pattern of CSEM estimates (Kolen et al., 1992; Brennan & Lee, 1999). Although test developers typically select the number of reporting scale score points to be slightly fewer than the number of raw score points to prevent test users from over-interpreting minor differences in scores, the choice of a reporting scale affects score reliability. Measurement error tends to increase as the number of scale score points for a particular test is reduced; scores converted to a relatively coarse reporting scale tend to have a larger standard error of measurement than scores reported on a more precise scale (Kolen et al., 1992). For scale scores obtained by nonlinear transformation, the CSEM estimated by any reasonable method should be amplified by the slope of the transformation function, showing variability that mirrors the variability in the slope of the transformation function across the score range. In a simulation study comparing three CSEM estimation methods, Lee, Brennan, and Kolen (1998, p. 19) found that, for all estimation methods tested, CSEM estimates exhibited only marginal bias around scale score values corresponding to inflection points in the transformation function, but tended to be underestimated where the slope of the transformation function was greatest and overestimated where the slope was least, although the average magnitude of this bias varied considerably depending on the estimation method used.

Although the literature comparing scale score CSEM estimation methods is fairly limited, previous studies have clarified the strengths and limitations of some methods, illuminating measurement conditions under which particular methods might be preferred. In simulation, CSEMs calculated using the Feldt-Qualls procedure exhibited larger sampling error variance than those estimated by any of the three direct methods, as well as higher average bias, except when assumptions of one of the other methods was clearly violated by the simulation conditions (Lee, Brennan, & Kolen, 1998, 2000). Because CSEM functions produced by the Feldt-Qualls procedure also tend to be relatively jagged (Brennan & Lee, 1999), use of this procedure is typically not recommended.

The most detailed comparison of scale score CSEM estimation methods is the Lee, Brennan, and Kolen (1998) simulation study. The study compared performance of the Feldt-Qualls, binomial error model, compound binomial error model, and IRT scale score CSEM estimation methods, computing the standard error of measurement for individual examinees under two types of conditions: unidimensional and multidimensional, and three different scale transformations. When examinee true scores were drawn from a two-dimensional distribution, with various correlations between the two ability dimensions, CSEMs estimated by the compound binomial error model method, which assumes test items are categorized into distinct content strata, showed less average bias than those estimated by the binomial error model method, which assumes test items are drawn from a single undifferentiated domain. However, when item response data was generated to be unidimensional, there was little difference in CSEMs from the binomial or compound binomial error model methods. One result of this study was that CSEMs estimated by the IRT direct method exhibited the least bias and lowest sampling error variance in not only all the unidimensional, but also all the multidimensional, item response conditions. The IRT method allows items to differ in difficulties and discriminations, but the binomial and compound binomial methods require either equal item difficulties (binomial method) or equal within strata item difficulties (compound binomial method) along with correspondingly equal item variances.

Delta method CSEM approximation using a polynomial to approximate the function *g*, two variants of which are used in this study, has been included in only two prior comparison studies, in both cases incorporating Feldt's (1984) estimate of raw score error variance for stratified tests. The polynomial version of the delta method has been shown to produce scale score CSEMs similar in magnitude to those generated by the Feldt-Qualls procedure (Feldt & Qualls, 1998) and the binomial error model method (Brennan & Lee, 1999), although slightly less smooth than those from the latter two methods. Similar findings have been obtained for comparison of CSEM estimates from the delta method, Feldt-Qualls procedure, and compound binomial error model method when all three methods relied on compound binomial error assumptions (Brennan & Lee, 1999). Review of the literature indicates that the four-parameter BCBM method, the third CSEM estimation procedure to be compared in this study, has not appeared in any previous published comparison studies of the delta method.

The few groupwise comparisons of scale score CSEM estimation or approximation methods that have been conducted for different tests and score scale types indicate the methods tend to yield relatively similar CSEM functions, with only the Feldt-Qualls approximation procedure consistently producing larger, more variable CSEM estimates. Given the similarity of performance among the scale score CSEM estimation and approximation methods, the preferred method in a particular measurement situation will depend on available computer resources and sample size, as well as the assumptions an investigator is willing to make about the process that generated the observed test score data. Because the computational burden for procedures estimating the CSEM directly from individual scale scores is relatively high, as are sample size requirements for the IRT-based direct procedure (Kolen, Zeng, & Hanson, 1996), approximate methods of scale score CSEM estimation may be necessary, particularly for applications involving small sample sizes, for example, analysis of field test data in the early phases of instrument development. Using a polynomial to approximate the raw-to-scale score conversion table values for the four ACT tests, this study compares scale score CSEMs obtained by the delta method approximation under the CTT and IRT models to CSEMs estimated under a four parameter BCBM method using the exact score transformation.

The Current Comparison of Three Methods

Three methods for computing CSEMs, the CTT delta method, the IRT delta method, and the four-parameter BCBM method are compared using real test data from approximately 445,000 examinees who took the four ACT subject-area tests: English (75 items), mathematics (60 items), reading (40 items), and science (40 items).

The previous scaling (Kolen et al., 1992) of each one of these four multiple choice tests was conducted separately by sequential application of a series of transformations, including variance stabilization, equipercentile equating, rounding, and truncation. Each scaling produced a raw score to scale score conversion where the scale scores range from 1 to 36 (ACT, 2007). This sequence of nonlinear transformations was represented as a single discrete transformation function in the form of a unique raw-score-to-scale-score conversion table for each test. In addition, each unique form of each subject area test had its own conversion table. The data in this study consists of a single distinct form for each one of the four subject-area tests.

Producing delta-method CSEM estimates under any set of assumptions requires first finding the slope of the transformation function at each raw score scale point. Regressions of the scale score values from each test's conversion table on a polynomial function of the corresponding raw score values, computed using the software JMP, produced continuous approximations to the transformation function for each of the four tests. Choosing the degree of the polynomial approximation for each conversion table involves subjective judgment. However, the magnitude of CSEM estimates across the score range does not depend greatly on the degree of the polynomial selected (Brennan & Lee, 1999, p. 14). To select a polynomial of the lowest possible degree that produced a smooth but adequate approximation to the score conversion table points, plots of the fitted polynomials and residuals, as well as R^2 and root mean square error fit index values, were inspected. Figures A-1 through A-4 in the Appendix plot the score conversion table values, as well as the polynomial used to approximate the conversion table, for each test. Sixth-degree polynomials were selected to fit the English, Math, and Science test conversion table values, but a fourth-degree polynomial was judged to provide a satisfactory approximation to the tabled conversion values for the Reading test, which showed the least departure from linearity. Because the conversion tables specified many-to-one transformations of multiple raw score values to single scale score points, for simplicity, the polynomials used for computation were fit to the means of the set of raw score values corresponding to each scale score point. The forms of the polynomials fit to the averaged raw score points differed little from those of the polynomials fit to all raw score points.

After selecting a polynomial to approximate each test's conversion table, the polynomial was differentiated, and its slope at each (average) raw score point obtained as the derivative. Since the same conversion table was used for both the CTT- and IRT-based delta method procedures, the same slope values appeared in both Equations 3 and 6, for each raw score point. It should be noted, however, that although the "raw scores" in each conversion table took the same values for both delta method procedures, in fact, they represented true score estimates obtained under different assumptions: for CTT, the observed number-correct scores in Equation 3, and for IRT, the model-estimated number-correct scores in Equation 6.

To generate raw score conditional measurement error variance estimates for the ACT tests under CTT assumptions, Thorndike's (1951) difference method was applied. For each subject-area test, item subsets for the difference method were obtained by an odd-even split of the items, and number-correct scores from each generated half-test were taken as estimates of the population quantities X_1 and X_2 in Equation 2. (The 75 item English test had a 38-37 odd-even item split.) Near-equivalence of the mean and variance for each half-test pair over the sample was verified, such that the assumption of classically-parallel half-tests in the population seemed reasonable. Obtaining the raw score error variance estimates by Equation 2, squared slope values from the polynomial approximation were then used to scale the raw score error variance, as shown in Equation 3, and the square root of the resulting quantity was taken as a CTT-based estimate of the scale score CSEM.

Implementing Lord's (1980) IRT method, item parameters were estimated under a three-parameter logistic IRT model from the item response data using BILOG-MG 3 (Zimowski, Muraki,, Mislevy, & Bock, 2002). A Newton-Raphson method was used to locate pairs of corresponding estimated number-correct and theta scores, $(\hat{\xi}_0, \hat{\theta}_0)$, for which the estimated number-correct score took on integer values. Substituting $\hat{\xi}_0$ values corresponding to the average raw score values in the conversion tables into the first term of Equation 5, and matching $\hat{\theta}_0$ values as well as item parameter estimates into the second term, provided estimates of the raw score conditional error variance. IRT-based scale score CSEM estimates were computed using Equation 6 as the square root of the resultant scale score error variance.

The four-parameter BCBM method was estimated using data from each subject-area test following the procedure detailed by Hanson (1991). The model requires a fixed value for raw score reliability, so stratified alpha coefficients estimated from each test's sample data, given stratum indicators drawn from the tests' tables of specifications, were used as reliability values. Estimates for conditional probabilities and true scores from the four-parameter beta compound binomial model were substituted into Equation 8, with the subject-area-specific score conversion table used for the other two methods again treated as the transformation. For each raw score point, $\hat{\tau}_0$, the square root of the equation's solution provided a BCBM-based scale score CSEM estimate, a third estimated CSEM to be compared to the two delta-method-based estimates obtained previously. Since CSEMs are typically represented as smooth functions (e.g., Kolen, Zeng, & Hanson, 1996, p. 137), cubic spline functions were fit to the discrete CSEM estimates generated at specific estimated true scale score values by the three methods. To allow comparison among methods, a .10 smoothing parameter value, selected by generalized crossvalidation to fit the BCBM English CSEM estimates, which appeared to be among the most regular sets of estimates, was used to smooth all plots.

Results

Graphs of the scale score CSEMs computed by the three methods for each ACT subject-area test are displayed in Figures A-5 through A-8 in the Appendix . Prior to spline smoothing, the discrete CSEM values for all procedures appeared to follow relatively smooth curves, although the patterns for the CTT-based delta method CSEM estimates were somewhat less smooth than those for the other two methods. CSEM estimates for scale scores at the low end of the scale score scale are not included in the figures because their estimates were very unstable due to the small number of examinees with such scores. The CSEM estimates for the ACT English test are very similar for all three methods except for a bump around 18 for the BCBM. The estimates tend to vary closely around 1.5 in the middle of the scale score scale, but decrease more sharply to 1 or below at the bottom and top of the scale score scale. The CSEM estimates for the ACT Mathematics test are also very similar for all three methods, although the BCBM method's curve is bumpier. Similar to English, but with more variability, the Mathematics CSEM estimates also tend to vary closely around 1.5 in the middle of the score scale, but then diverge away from 1.5 at the lower and upper extremes of the scale score scale.

The three different methods' Reading CSEM estimates are all very similar across the entire score scale, although, again, the BCBM method's curve is bumpier. In the bottom and middle of the score scale the estimates are very close to but a little above 2.0, whereas at the top of the score scale they decrease dramatically. The CSEM estimates from the three methods are also very similar for Science although they do differ somewhat in the middle of the score scale where they reach a local minimum. At the low end of the score scale the Science CSEM estimates are about 2.25. They decrease in the middle of the score scale then increase to a local maximum before dramatically decreasing at the top of the score scale.

Discussion

Comparison of the results from the three CSEM estimation techniques indicates that the delta method approximation may be a useful procedure for obtaining scale score CSEM estimates under a variety of test score models. For the ACT test data used in this study, the approximate CSEM values produced by the two delta method procedures varied little from the estimates obtained from the four-parameter BCBM method which used the exact score transformation function. This result seems likely to hold for commonly-used score scale types because previous work suggests the relative magnitude of CSEM estimates from a particular set of estimation procedures tends to be fairly similar for typical score transformations (e.g., percentile ranks, grade-equivalent scores) (Brennan & Lee, 1999; Lee, Brennan, & Kolen, 1998), but may not generalize to all conceivable score scale types. However, the result that the three methods produced highly similar CSEM estimates can be loosely extended to data from other test instruments or examinee populations because it is consistent with the conclusions of other studies contrasting other sets of CSEM estimation methods (Brennan & Lee, 1999; Feldt & Qualls, 1998). The CSEMs obtained from the compared methods were highly similar, considering only empirical criteria, it would be difficult to justify recommending any one particular estimation method over another.

The CTT-based delta method allows CSEM estimation even when conditional raw score frequencies are very low, which may be perceived as advantageous for obtaining CSEMs of scores in the tails of a scale score distribution. However, previous authors (e.g., Feldt, Steffen, & Gupta, 1985), demonstrate that raw score error variance estimates obtained from Thorndike's

(1951) difference method can be unstable for small sample sizes, and they recommend smoothing the raw score CSEM estimates. This could be done before applying the delta method to obtain the scale score CSEM estimates.

Previous studies producing CSEMs by the delta method with a polynomial approximation (albeit using different raw score error variance estimators than either of those used in the present study) reported that these methods produced relatively jagged CSEMs (Brennan & Lee, 1999; Feldt & Qualls, 1998). Feldt and Qualls asserted that the jagged pattern reflected real irregularities in the scale score transformation function, but Brennan and Lee argued that irregularities in the estimated CSEMs were perhaps due to the over fit of the approximating polynomials, because population transformation functions should be smooth. The results of this study demonstrate that a jagged pattern is not always a feature of CSEM estimates produced by the delta method with a polynomial conversion approximation at least when sample sizes are large. Although the CSEM estimates from the CTT-based delta method were slightly more jagged than the estimates from the IRT-based delta method, the approximating polynomial was the same for both methods, thus the jaggedness of the CTT-based delta method CSEM estimates can be attributable to irregularities in the set of difference-method raw score error variance estimates. If the assumptions of all three estimation procedures were equally plausible, then adopting Brennan and Lee’s position (see also Lee, Brennan, & Kolen, 1998), the IRT-based delta method or BCBM method might be favored over the CTT-based delta method although smoothing of the raw score error variances could be used to remove any jaggedness.

Although the estimation methods compared in this study yielded similar CSEM estimates for scale scores from the four ACT tests, as emphasized in other authors’ conclusions (e.g., Lee, Brennan, & Kolen, 2000), choice of a scale score CSEM estimation method should depend on its assumptions and the types of measurement error expected to predominate in a particular measurement situation. Given test design information, and response data from an examinee population of interest, particular assumptions underlying each of the three methods might be judged unreasonable based on theory or statistical test results.

The CTT-based delta method appears to have the least stringent assumptions: uncorrelated true and error scores and the division of the test into parallel half-tests, but this simplicity is offset by the conditioning on observed raw score instead of true raw score, as made clear in Equation 2, and this conditioning leads to a negative correlation between the error scores on the two half-tests (Woodruff, 1990). However, as the results in this paper demonstrate, the method produces results similar to the other two methods used in this paper, and intuition suggests that the negative correlation between the two half-test error scores may be less of a problem as test length increases especially in the middle of the raw score scale.

The IRT-based delta method requires the usual unidimensional IRT model assumptions and the delta method assumptions, as well as an adequate sample size to estimate IRT item parameters. Unidimensional IRT models allow for only one "true" score in contrast to the BCBM that allows for several, albeit, a priori true scores. However, if content category scores for a certain test are highly correlated in the population, so that the item domains behave as undifferentiated, the CTT-based delta method could be an appropriate, simple mechanism to produce CSEM estimates. If, furthermore, the item response data is unidimensional, in a factorial sense, then the IRT-based delta method procedure could be used. See Lee, Brennan, and Kolen (1998, pp. 25-30) for a discussion that emphasizes the relationships between the assumptions of the various estimation procedures and the meanings of their estimated CSEMs. Another consideration for picking one method over another is theoretical and computational complexity as regards explaining the method to test score users.

The four parameter BCBM assumes that items are randomly sampled from a population of items that are divided into different strata, and each stratum has its own true score. The crucial assumption is that given a particular stratum true score value, the observed raw score for that stratum has a binomial distribution. This is true when, conditional on true score, the observed stratum score is a sum of the independent Bernoulli distributed item score random variables all of which have equal probabilities of success. Various assumptions about the random sampling of items and examinees can be used to justify a binomial model even when items are not all equally difficult for an examinee with a given true score. The model also may be robust to violations of equal conditional item difficulties because the other models do not make this assumption and all three models yield very similar results. That the three methods produce similar results even though they are based on different assumptions and approximations suggests that although their underlying assumptions may not be completely realistic in practice, their application to real data can produce consistent and useful results.

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Appendix

Figures A-1 -- A-8

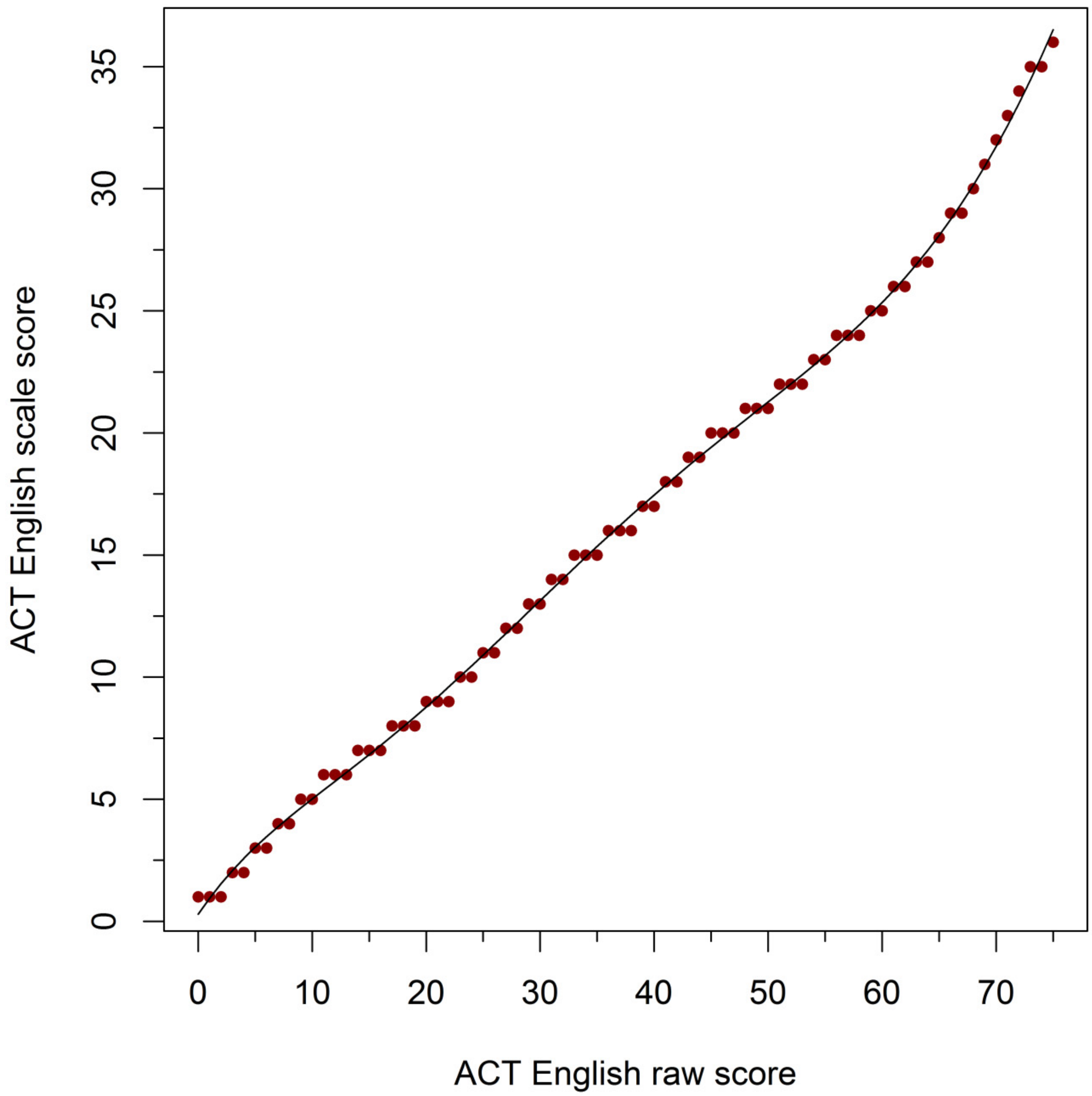

Figure A-1. Sixth-degree polynomial function fit of raw to scale score conversion table values for the selected form of the ACT English Test

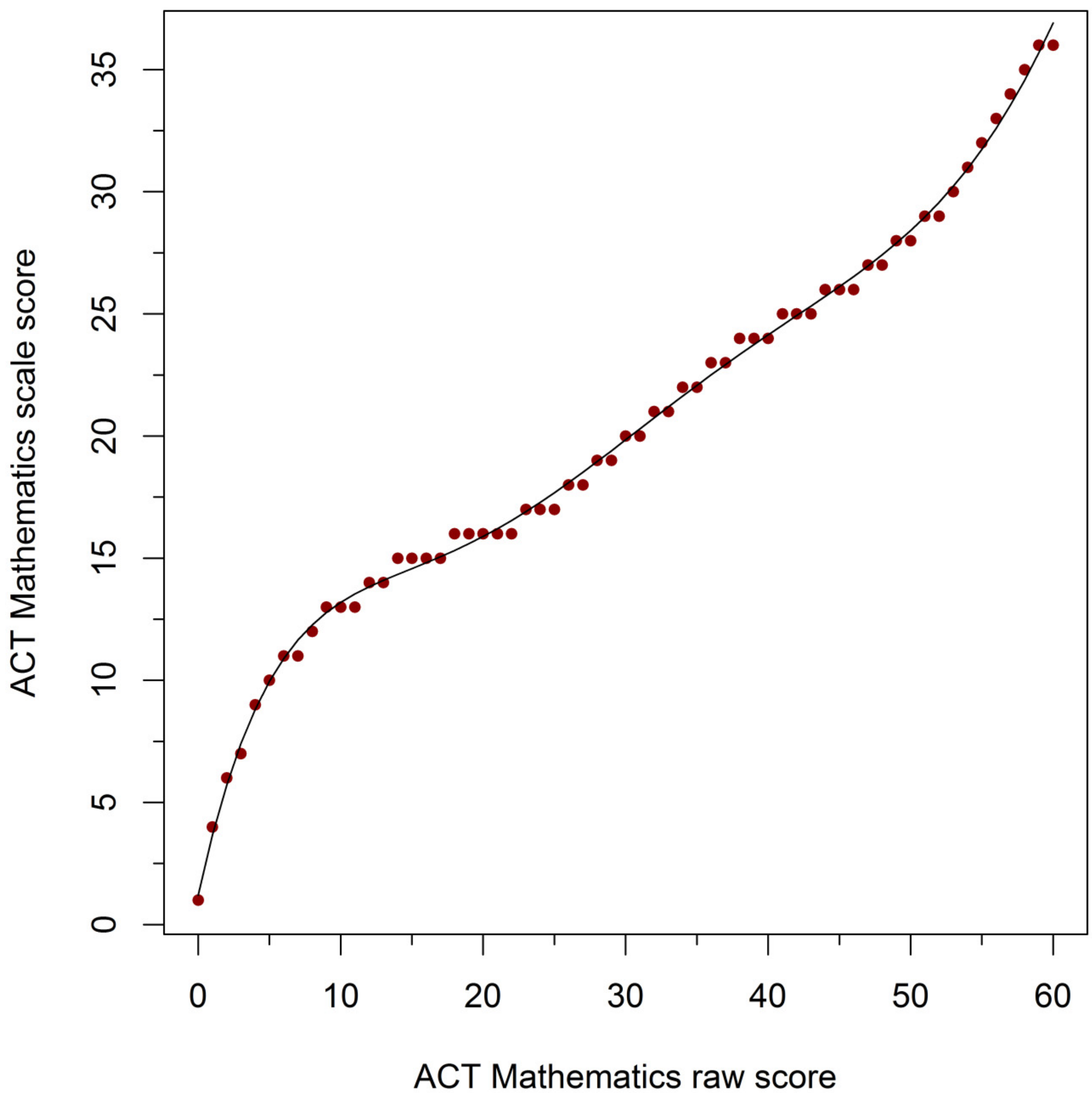

Figure A-2. Sixth-degree polynomial function fit of raw to scale score conversion table values for the selected form of the ACT Mathematics Test

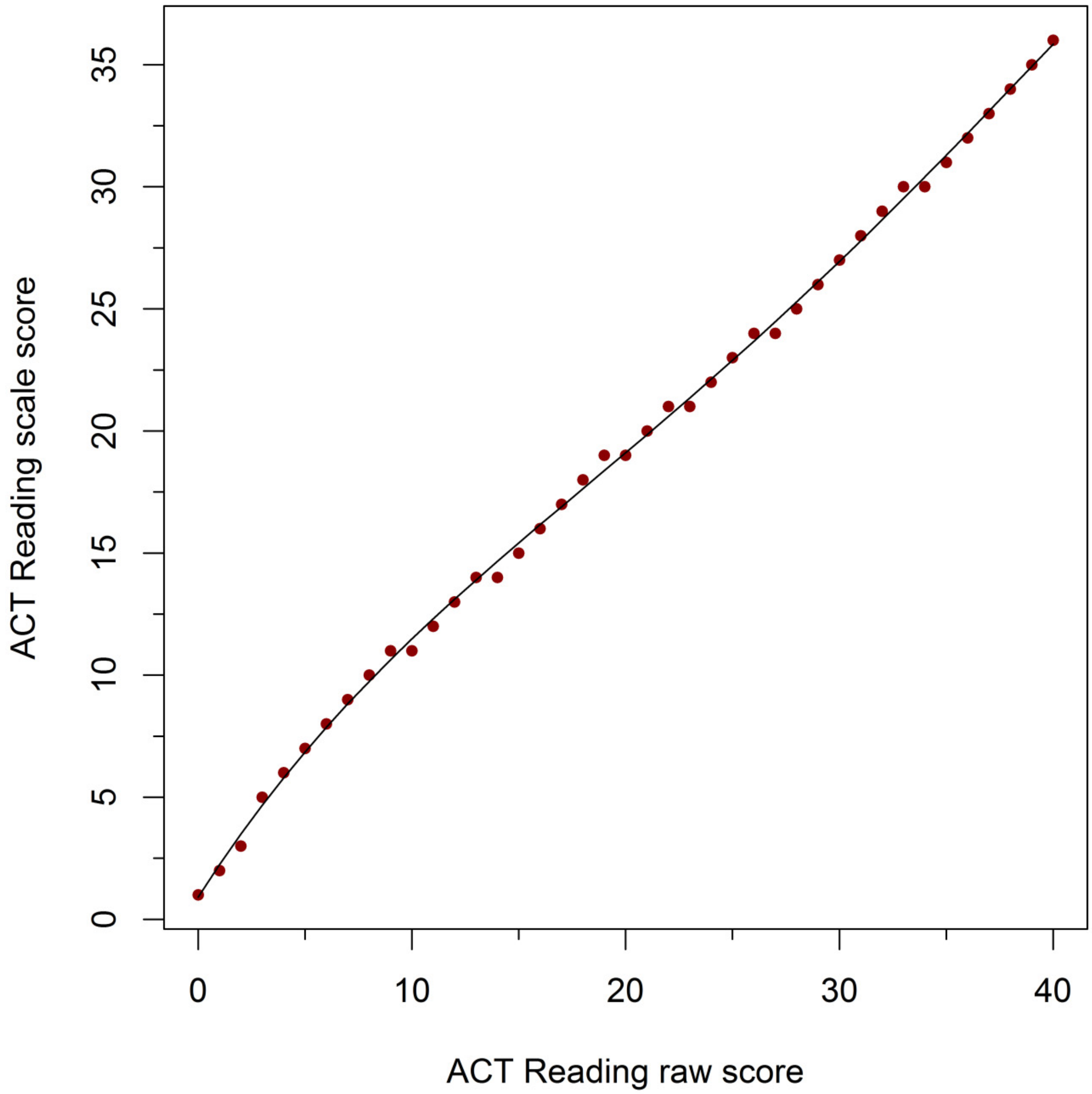

Figure A-3. Fourth-degree polynomial function fit of raw to scale score conversion table values for the selected form of the ACT Reading Test

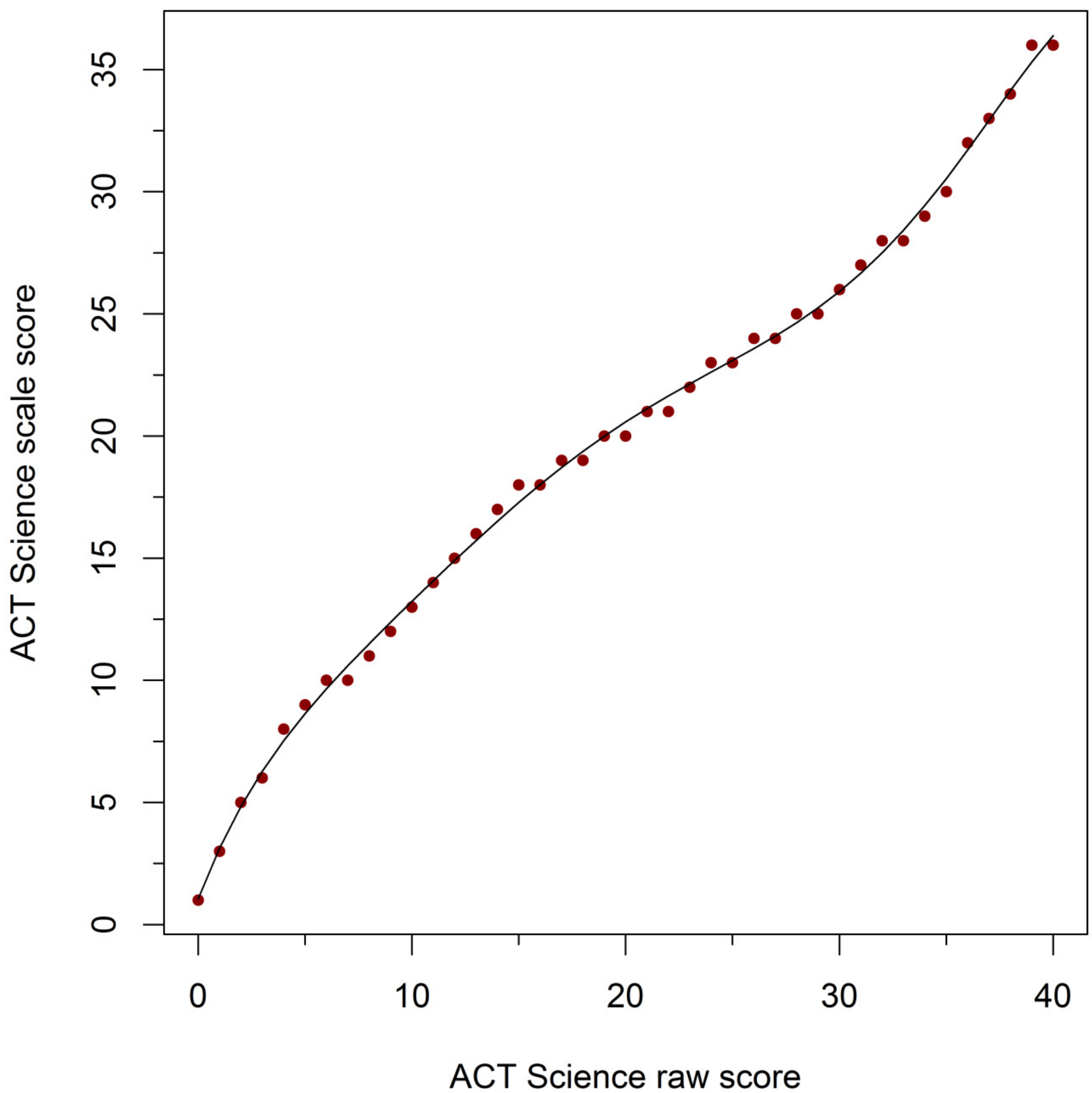

Figure A-4. Sixth-degree polynomial function fit of raw to scale score conversion table values for the selected form of the ACT Science Test

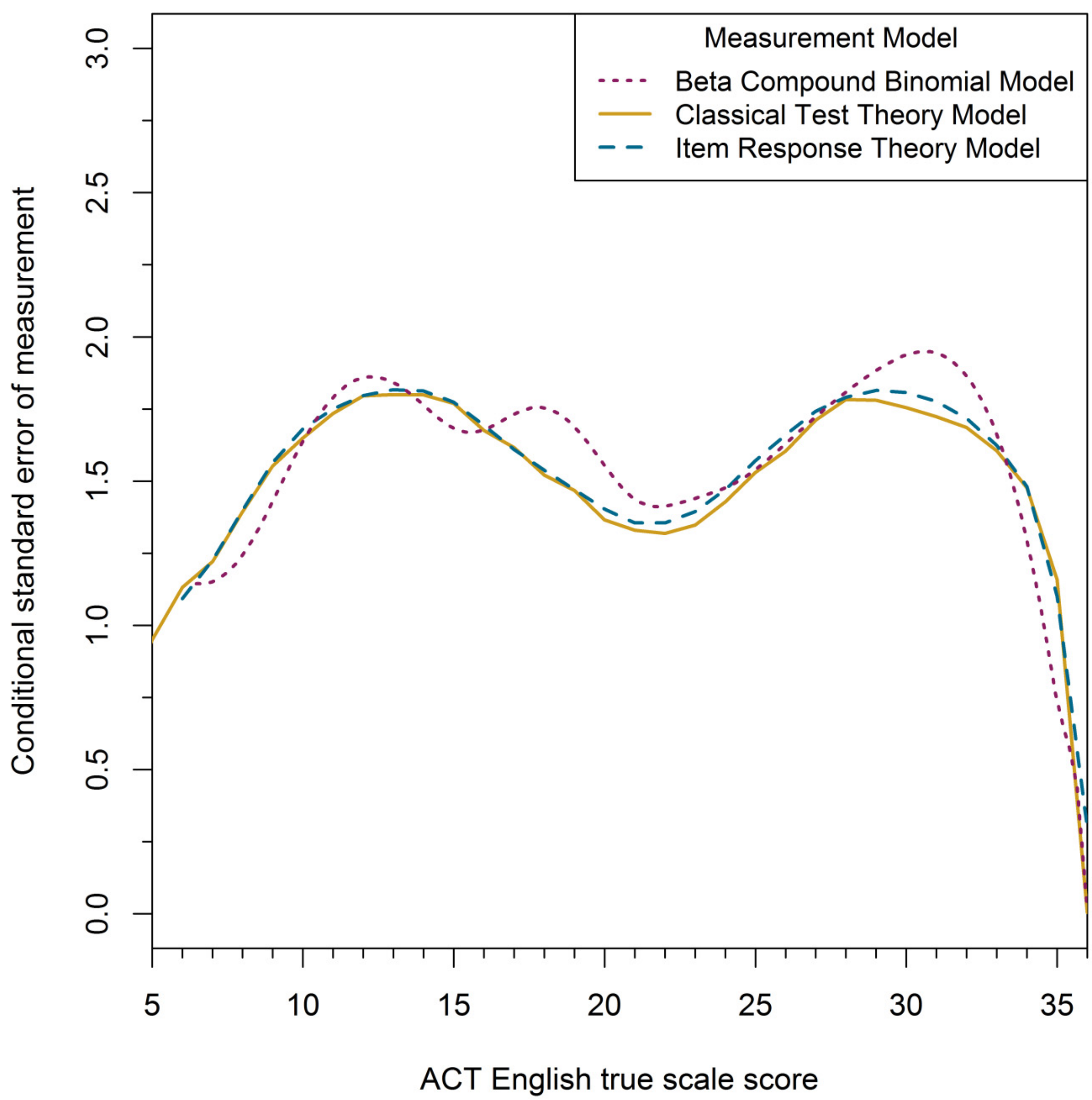

Figure A-5. Estimated conditional standard error of measurement for observed scale score given true scale score for the ACT English Test form

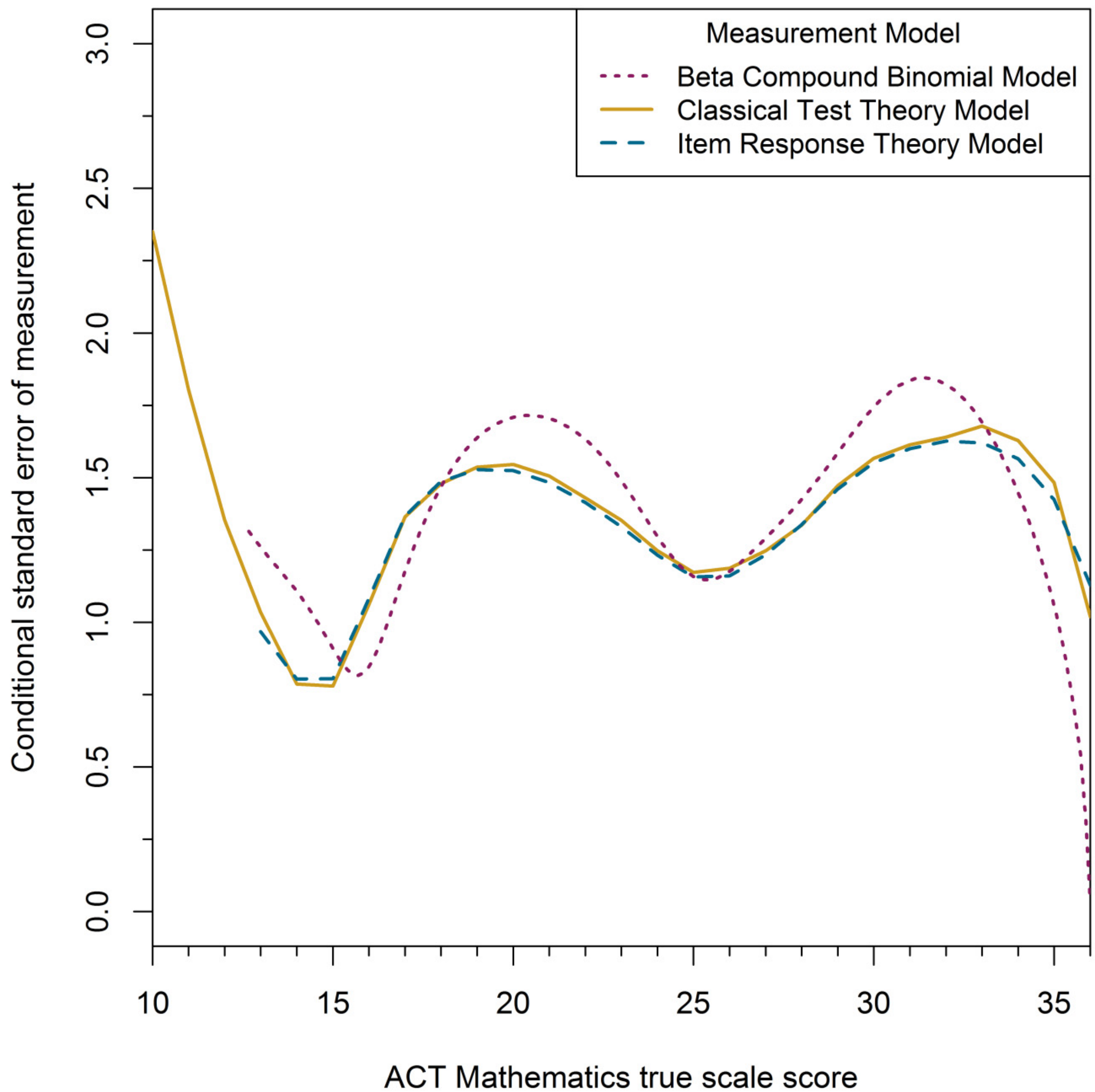

Figure A-6. Estimated conditional standard error of measurement for observed scale score given true scale score for the ACT Mathematics Test form

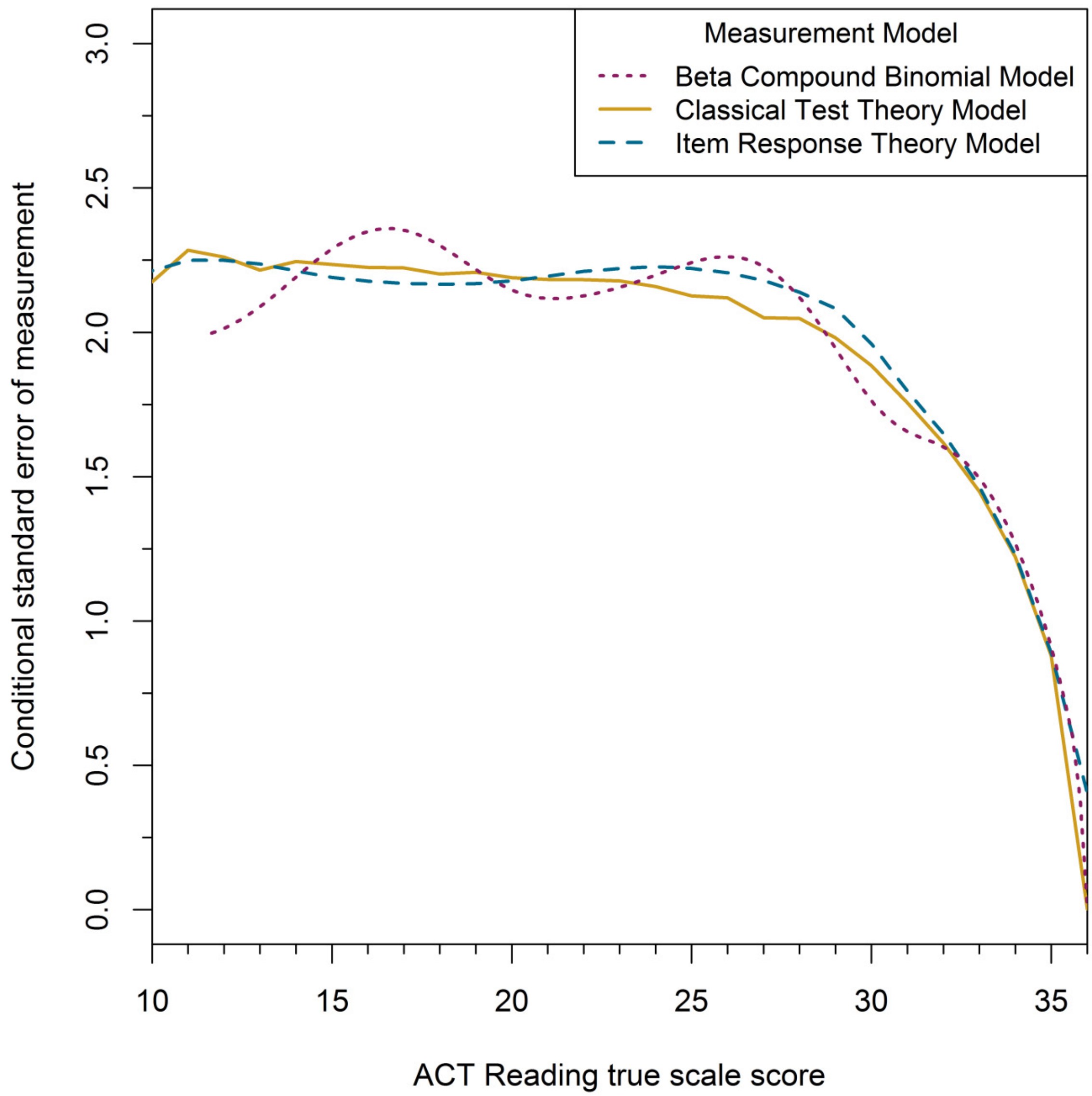

Figure A-7. Estimated conditional standard error of measurement for observed scale score given true scale score for the ACT Reading Test form

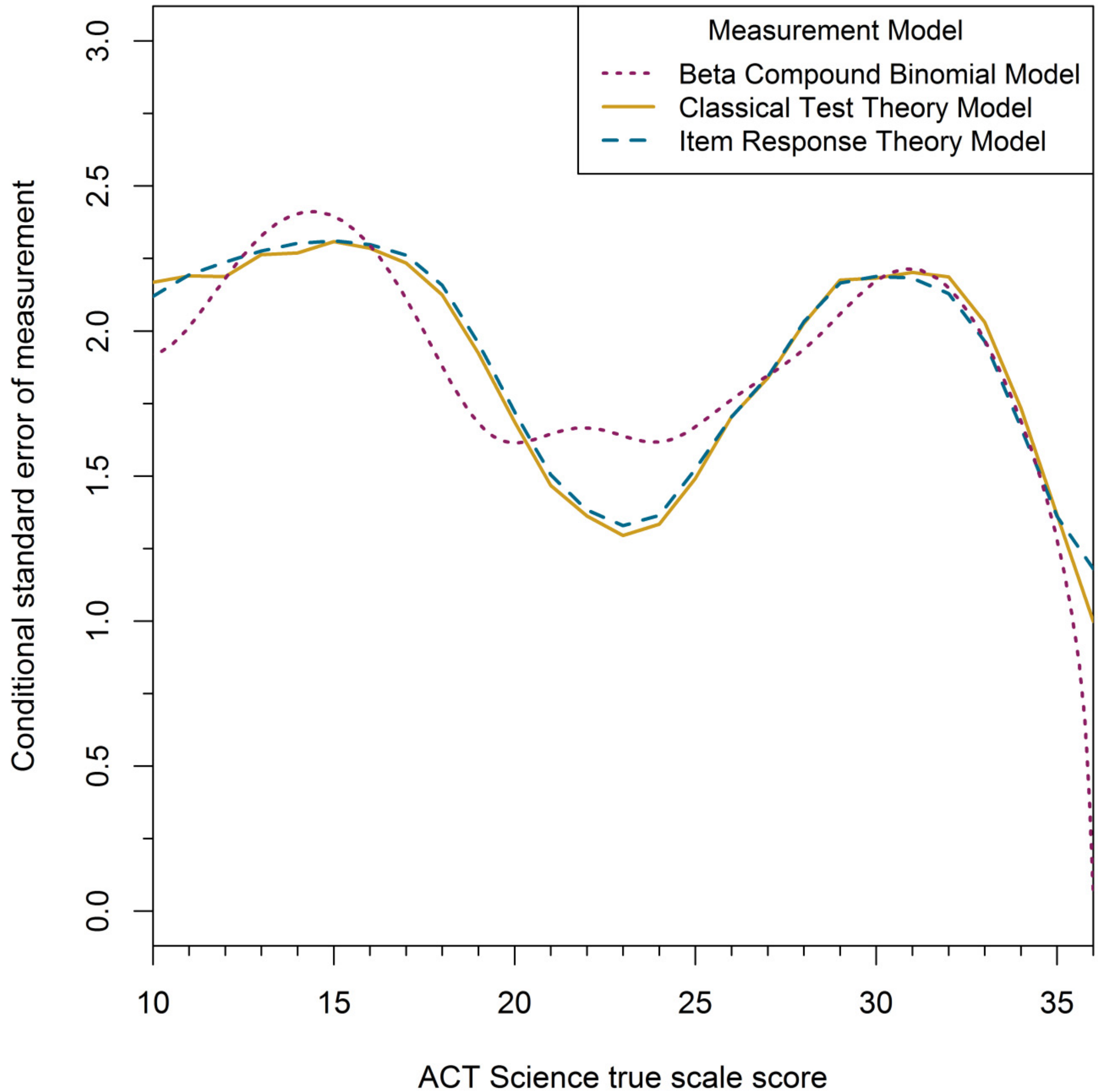

Figure A-8. Estimated conditional standard error of measurement for observed scale score given true scale score for the ACT Science Test form

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